so that the water may spout through it to the greatest horizontal distance on the plane. What difference in the result when the cylinder stands on a shelf of known height above the plane?

SOLUTION BY J. B. SMITH, Hampden-Sidney, Va.

Let a be the height of the cylinder and b the height of the orifice above the plane. Let v_0 be the velocity of the jet at the orifice. Then

$$v_0 = \sqrt{2g(a-b)}, \quad x_0 = v_0 t, \quad \text{and} \quad y = b - \tfrac{1}{2}gt^2.$$

Hence,

$$y = b - \frac{x^2}{4(a-b)}$$

is the equation of the trajectory, the origin being at the base of the cylinder and the y-axis being the vertical through the orifice.

For $y = 0$, we have $x^2 = 4b(a-b)$. As x^2 is a maximum when x is, we have

$$\frac{d(x^2)}{db} = 4(a-b) - 4b = 0,$$

$$\therefore\ b = a/2.$$

Hence the range will be a maximum when the height of the orifice is half that of the cylinder.

If the cylinder stands on a shelf of height h above the plane, we wish to find the maximum range on the line $y = -h$. In this case we have

$$x^2 = 4(b+h)(a-b).$$

It follows that the range is a maximum when $b = (a-h)/2$. If $h = a$, the orifice must be at the bottom of the cylinder; if $h > a$, b would be negative, but as this is impossible, the greatest range for the given cylinder is obtained by making the orifice at the bottom.

Also solved by HORACE OLSON and B. LIBBY.

NUMBER THEORY.

200. Proposed by R. D. CARMICHAEL, Indiana University.

Find the general solution, in relatively prime integers, of the equation $x^2 + y^2 = z^4$.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

Let $x = p^2 - q^2$, $y = 2pq$; then

$$x^2 + y^2 = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2 = z^4.$$

Let $p = r^2 - s^2$, $q = 2rs$; then

$$p^2 + q^2 = (r^2 - s^2)^2 + (2rs)^2 = (r^2 + s^2)^2;$$

therefore

$$x^2 + y^2 = (p^2 + q^2)^2 = (r^2 + s^2)^4 = z^4.$$

Retracing,

$$x = (r^2 - s^2)^2 - (2rs)^2, \qquad y = 4rs(r^2 - s^2), \qquad z = r^2 + s^2;$$

therefore

$$[(r^2 - s^2)^2 - (2rs)^2]^2 + [4rs(r^2 - s^2)]^2 = (r^2 + s^2)^4,$$

where r and s may be any numbers prime to each other, one odd and the other even.

Let $r = 2$, $s = 1$, and we have

$$7^2 + 24^2 = 625 = 25^2 = 5^4.$$

Let $r = 3$, $s = 2$; then we get

$$119^2 + 120^2 = 169^2 = 13^4.$$

Let $r = 4$, $s = 1$, and we find

$$161^2 + 240^2 = 289^2 = 17^4.$$

203. Proposed by R. D. CARMICHAEL, Indiana University.

Find solutions in integers of the equation

$$2x^2 + 1 = 3y^2. \tag{1}$$

I. SOLUTION BY E. E. WHITFORD, New York City.

Let $2x = z$. Then

$$z^2 - 6y^2 = -2. \tag{2}$$

By inspection the solution in smallest positive integers is $z_1 = 2$, $y_1 = 1$. The problem will not lose in generality if the solution be limited to positive integers. To represent concisely all the positive solutions without exception and without repetition I have derived the following formula:

$$z + y\sqrt{6} = \frac{(z_1 + y_1\sqrt{6})^{2k+1}}{2^k},$$

where

$$z_1 = 2, \qquad y_1 = 1, \qquad k = 0, 1, 2, 3, \cdots.$$

Therefore

$$z = 2, \quad 22, \quad 218, \quad 2158, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad \cdots.$$

Since the values of z are even each set gives a solution of equation (1).

$$x = 1, \quad 11, \quad 109, \quad 1079, \quad 10681, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad 8721, \quad \cdots.$$

These results were obtained by Euler in his "Algebra," by de la Roche in his "Larismetique" (1520), who copied from the "Triparty" of Chuquet (1484); and probably by nearly everyone who tried, before decimal fractions came into common use, to find the approximate value of the square root of 6.

Equation (2) is a slightly generalized form of the Pell equation $x^2 - Ay^2 = 1$.